1. Revisiting Kepler's measurements

Kepler's first law states that the planets revolve around the sun in an elliptical pattern with the sun as one of the ellipse's foci. Earlier, we defined an astronomical unit (AU) as the distance between the earth and the sun. However, since earth revolves in an elliptical pattern, its distance to the sun changes through its orbit. Let a be the distance of the earth's semimajor axis, and let e be its linear eccentricity. As the sun is on of the earths focal points, the furthest it every gets from the sun is $a + e$. We say Earth is in *aphelion* when it is furthest from the sun. The closest it gets to the sun is $a - e$. We say Earth is in *perhelion* when it is closest to the sun. Therefore, the earth's average distance from the sun is

$$\frac{(a+e)+(a-e)}{2} = \frac{2a}{2} = a.$$

So its average distance to the sun is the same as the length of its semimajor axis. From now on, the AU will refer to the length of the earth's semimajor axis. It wasn't until the 19th century that the AU was measured with any sort of remarkable accuracy. The most precise modern measurements put $AU \approx 92,955,807.2$ miles.

Let's analyze some of Kepler's measurements of Mars' orbital pattern. Refer to table 4.2 in your text for reference. Kepler found that Mars' furthest distance from the sun is about 1.6678 AU and its smallest distance is about 1.3850 AU. Therefore, its average distance is about $(1.6678+1.3850)/2 = 1.5264$ AU. Thus, the semimajor axis of Mars' orbit has length $a = 1.5264$ AU. We also see that $a + e = 1.6678$ AU and $a - e = 1.3850$ AU, so its linear eccentricity is about $0.2828/2$ AU $= 0.1414$ AU. This gives us that Mars' astronomical eccentricity is $\varepsilon = \frac{0.1414}{1.5264} \approx 0.0926$.

Notice that Mars' (as well as all the other planets') astronomical eccentricity is very small. This means that it does not have much "flatness" or that its orbit is nearly circular. This fact is the reason why Copernicus' model, while erroneous, was still very accurate in its measurements. Of all the planets (the six known at the time of Kepler), Mercury's is the most "elliptical" in shape. Its astronomical eccentricity is about 0.2056.

2. Kepler and Mars

We will use the modern astronomical measurements in this section. The methods are still true to Kepler though. As noted above, 1 AU $\approx 92,955,807.2$ miles. Since Mars is on average 1.5237 AU from the sun, its semimajor axis is $a \approx (1.5237)(92,960,000) \approx 141.64$ million miles. The linear eccentricity is $e = \varepsilon \cdot a$. So the length of the semiminor axis is

$$b = \sqrt{a^2 - e^2} = \sqrt{a^2 - \varepsilon^2 a^2}$$

$$= a\sqrt{1-(0.0934)^2} = a\sqrt{1-0.0087}$$

$$= a\sqrt{0.9913} = 0.9956 \cdot a.$$

So $b = 0.9956)(141.64) = 141.02$ million miles. We can now compute the maximum and minimum distances of Mars from the sun.

$a + e = a(1+\varepsilon) = a(1+0.0934) = 1.0934a = (1.0934)(141.64) \approx 157.87$ million miles.

$a - e = a(1-\varepsilon) = a(1-0.0934) = 0.9066a = (0.9066)(141.64) \approx 128.41$ million miles.

Now let's look at some of the earth's measurements. The semimajor axis is of course 1 AU. Its astronomical eccentricity is about $\varepsilon \approx 0.0167$. So its semiminor axis is about

$$b = \sqrt{a^2 - \varepsilon^2 a^2} = \sqrt{1-(0.0167)^2} = \sqrt{1-0.0003} = \sqrt{0.9997} \approx 0.9998AU.$$

The minimum distance of the earth from the sun is

$$a - e = a - \varepsilon a = 1 - 0.0167 = 0.9833AU.$$

Its maximum distance from the sun is

$$a + e = 1.0167AU.$$

So the earth's orbit varies from about 91.41 million miles from the sun to about 94.51 million miles.

3. Kepler's Equation

Given the data in table 4.2, we can calculate many of the characteristics of the planets' orbits. However, what about determining the actual position of a planet at any given time t? The planets move in an elliptical orbit, and Kepler's second law says that wach planet sweeps in an equal area over any time interval. So it seems feasible that we can find a planet's position in terms of time.

Consider Figure 1 below. We have an ellipse with semimajor axis a and foci at $(-e, 0)$ and $(e, 0)$. Say the sun S is the focus $(e, 0)$. The points N and A will represent the perhelion and aphelion respectively. The point P represents a the planet's position. r is the distance of the planet P from the sun S. Let α be the angle from the major axis to the line $\overline{PS}$ as shown. Say the planet's orbital path is given by $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$.

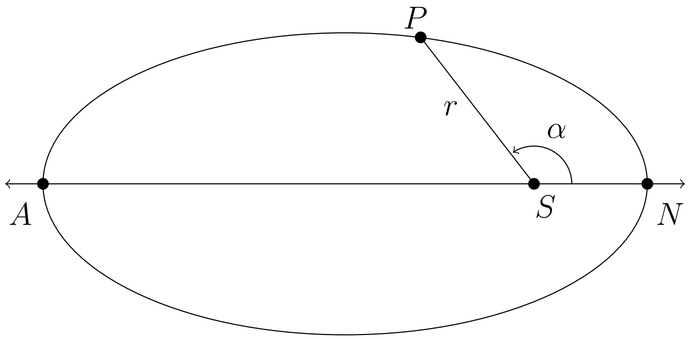

Figure 1

Now consider figure 2 below, which is just an augmentation of figure 1. We superimpose our planet's orbit onto the circle with radius a as shown. If the planet's position is at (x, y), let $X = (x, 0)$, and let $P = (x, y_0)$ be the corresponding point on the circle. That is, $x^2 + y_0^2 = a^2$. Create a triangle XOP_0. Let β be the angle from the major axis to the line $\overline{OP_0}$ as shown.

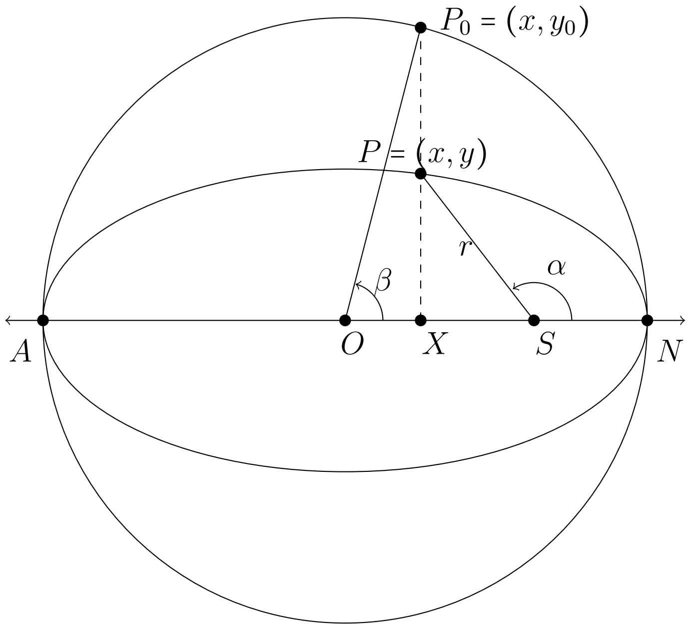

Figure 2

We will not give all the details of Kepler's argument, but you can find them in your text. Our first step is to express α and r in terms of the angle β. Notice that $\cos\beta = \frac{OX}{OP_0} = \frac{x}{a}$. From figure 2 we have

$$x = a\cos\beta.$$

Also note that $\sin\beta = \frac{y_0}{a}$. So $y_0 = a\sin\beta$. Since $y = \frac{b}{a}y_0$. Thus,

$$y = b\sin\beta.$$

Let's consider triangle ΔSPX. $SX = e - x$. Also recall that $e^2 = a^2 - b^2$ and $\sin^2\beta + \cos^2\beta = 1$. Using the Pythagorean theorem, Kepler calculated as follows :

$$\begin{aligned}
r^2 &= y^2 + (e-x)^2 \\
&= b^2\sin^2\beta + (e - a\cos\beta)^2 \\
&= b^2\sin^2\beta + a^2\cos^2\beta - 2ae\cos\beta + e^2 \\
&= (a^2 - e^2)\sin^2\beta + a^2\cos^2\beta - 2ae\cos\beta + e^2 \\
&= a^2 - e^2\sin^2\beta - 2ae\cos\beta + e^2 \\
&= a^2 + e^2(1 - sin^2\beta) - 2ae\cos\beta \\
&= a^2 + e^2\cos^2\beta - 2ae\cos\beta \\
&= (a - e\cos\beta)^2
\end{aligned}$$

Since, $a > e \geq \cos\beta$, $a - e\cos\beta > 0$. $r > 0$, so we can take the roots of both sides above to get $r = a - e\cos\beta$. We now factor the a on the right side to get $r = a(1 - \frac{e}{a}\cos\beta)$. Thus we have

$$\boxed{r = a(1 - \varepsilon\cos\beta)}$$

So we have determined the value of r in terms of β.

Consider triangle ΔSPX. We have

$$\tan\alpha = -\tan(\pi - \alpha) = -\frac{y}{e-x} = \frac{b\sin\beta}{a\cos\beta - e}.$$

Using some trigonometry, we can derive the following :

$$\boxed{\tan\frac{\alpha}{2} = \sqrt{\frac{1+\varepsilon}{1-\varepsilon}}\tan\frac{\beta}{2}}$$

The above formula is known as *Gauss' formula*, for the German mathematician Carl Gauss. Notic that Gauss' equation gives us α in terms of β. So we have completed our first goal.

Our second goal is to express β in terms of time t. Then we will have α and r in terms of time, and thus, the position of the planet in terms of time. Refer to figure 2 again. We have that *Area section* P_0XN = *Area sector* P_0ON- *Area* ΔP_0XO. P_0ON is a circular section with radius a and angle β. Therefore, the area is $\frac{1}{2}a^2\beta$. Of course, the area of the triangle ΔP_0XO is $\frac{1}{2}xy_0$, where $y_0 = a\sin\beta$. So the area of section P_0XN is

$$P_0XN = \frac{1}{2}\beta a^2 - \frac{1}{2}xa\sin\beta.$$

From the previous section we have that the area of the elliptical section PXN is $\frac{b}{a}\left(\frac{1}{2}\beta a^2 - \frac{1}{2}xa\sin\beta\right)$

$$= \frac{1}{2}\beta ab - \frac{1}{2}xb\sin\beta.$$

Let A_t be the area swept by the planet as it moves from the perhelion N to the point P. Then the area A_t is $PXN-$Area of ΔPXS.

$$\begin{aligned}
A_t &= \left(\tfrac{1}{2}\beta ab - \tfrac{1}{2}xb\sin\beta\right) - \tfrac{1}{2}(e-x)y \\
&= \left(\tfrac{1}{2}\beta ab - \tfrac{1}{2}xb\sin\beta\right) - \tfrac{1}{2}(e-x)b\sin\beta \\
&= \tfrac{1}{2}\beta ba - \tfrac{1}{2}eb\sin\beta \\
&= \tfrac{1}{2}\beta ab - \tfrac{1}{2}\varepsilon ab\sin\beta \\
&= \tfrac{ab}{2}(\beta - \varepsilon\sin\beta)
\end{aligned}$$

Recall that Kepler's second law states that $\frac{A_t}{t}$ is constant for any time t. So in particular it is this constant for $t = T$, where T is the period of the planet's orbital period. Since the area of an ellipse is $\frac{A_t}{t} = \frac{ab\pi}{T}$. Thus, $A_t = \frac{ab\pi t}{T} = \frac{ab}{2}(\beta - \varepsilon\sin\beta)$. After simplification we get,

$$\boxed{\beta - \varepsilon \sin \beta = \frac{2\pi t}{T}}.$$

The equation above is known as *Kepler's equation.* Notice that it expresses the angle β in terms of time t. Thus, we can express α and r in terms of t. Following Kepler's method, we have found a relationship between a planet's position in terms of time, meaning given a particular time t, we can determine a planet's position P.